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Summaries of the lectures by:

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Relations between interior and exterior boundary value problems

by

M.M. Schiffer

In many problems of applied mathematics one has to solve a boundary value problem for an elliptic partial differential equation for the exterior $\tilde{D}$ of a surface S while the corresponding solution for the interior D of S is trivially known.

Example: Let S be the surface of a body immersed into a steady irrotational flow in the x-direction of an incompressible fluid. The velocity potential has the form $\phi = x - \varphi$ where φ is a regular harmonic function in $\tilde{D}$ with the normal derivative $\frac{\partial\varphi}{\partial n} = \frac{\partial x}{\partial n}$ on S. One asks, in particular, for the virtual mass in the x-direction of the body, defined as $W_x = \iiint_{\tilde{D}} (\nabla\varphi)^2 d\tau$. The same boundary value problem with respect to D would yield $\varphi \equiv x$. The purpose of the paper is to show how the knowledge of the interior solution helps to solve the exterior problem.

We introduce the linear function spaces Σ and $\tilde{\Sigma}$ of functions harmonic in the closure of D and $\tilde{D}$ and define the scalar products for any two functions $\alpha, \beta \in \Sigma$

$$[\alpha,\beta] = \frac{1}{4\pi}\int_S\int_S \frac{\partial\alpha(P)}{\partial n} \frac{\partial\beta(Q)}{\partial n} \frac{1}{r(P,Q)} d\sigma_P \, d\sigma_Q$$

$$(\alpha,\beta) = \iiint_D (\nabla\alpha \cdot \nabla\beta)\, d\tau$$

$$\{\alpha,\beta\} = (\alpha,\beta) - [\alpha,\beta]$$

and use analogous definitions for $\tilde{\alpha},\tilde{\beta} \in \tilde{\Sigma}$. These scalar products have all positive-definite norms. It can be shown that for function pairs $\alpha,\beta \in \Sigma$, $\tilde{\alpha},\tilde{\beta} \in \tilde{\Sigma}$ with the boundary solutions $\frac{\partial\alpha}{\partial n} = \frac{\partial\tilde{\alpha}}{\partial n}$ and $\beta = \tilde{\beta}$ the following equalities hold:

$$[\alpha,\alpha] = [\tilde{\alpha},\tilde{\alpha}], \quad \{\beta,\beta\} = \{\tilde{\beta},\tilde{\beta}\}, \quad [\alpha,\beta] = -\{\tilde{\alpha},\tilde{\beta}\}$$

$$\{\alpha,\beta\} = -[\tilde{\alpha},\tilde{\beta}], \quad (\alpha,\beta) = -(\tilde{\alpha},\tilde{\beta}).$$

From these equations and by use of the Schwarz inequality one derives the inequalities:

$$(\tilde{\alpha},\tilde{\alpha}) \geq \frac{[\alpha,\beta]^2}{\{\beta,\beta\}} + [\alpha,\alpha], \quad (\tilde{\beta},\tilde{\beta}) \geq \frac{\{\alpha,\beta\}^2}{[\alpha,\alpha]} + \{\beta,\beta\}.$$

The first inequality yields lower bounds for the Dirichlet integral with the unknown function $\tilde{\alpha}$ in terms of the known solution α and an arbitrary test function β. The second inequality gives an analogous estimate for $\tilde{\beta}$ using now α as the test function. Both inequalities may be used to establish a Rayleigh-Ritz procedure for solving Neumann or Dirichlet boundary value problems for $\tilde{D}$.

Specialization of the above inequalities leads to

$$(\tilde{\alpha},\tilde{\alpha}) \geq (\alpha,\alpha)\,\frac{[\alpha,\alpha]}{\{\alpha,\alpha\}} \text{ if } \frac{\partial\alpha}{\partial n} = \frac{\partial\tilde{\alpha}}{\partial n} \quad \text{on } S$$

$$(\tilde{\beta},\tilde{\beta}) \geq (\beta,\beta)\,\frac{\{\beta,\beta\}}{[\beta,\beta]} \text{ if } \beta = \tilde{\beta} \quad \text{on } S.$$

Both inequalities become equalities for a set of functions which are complete in Σ and $\tilde{\Sigma}$ and are closely related to the Poincaré-Fredholm eigen functions of the surface S.

The general result can be applied to prove the following conjecture of Pólya: Let $W_m = \frac{1}{3}(W_x+W_y+W_z)$ denote the average virtual mass of a body; then $W_m \geq \frac{1}{2}V^2$ where V is the volume of the body.

As a side result of the proof we obtain the theorem: The lowest non-trivial positive eigen value of the Poincaré-Fredholm integral equation for every surface S is less or equal to 3.

Equality holds in the case of the sphere. The theorem allows an appreciation of the convergence of the series solution for the integral equation solving the boundary value problems of potential theory.

Extension of the method to more general elliptic differential equations is obvious.

Singular solutions of the Neumann problem

by

R. Timman

1. Except for an additive constant the Neumann problem:
Determine a solution Φ of $\Delta\Phi=0$, which satisfies $\frac{\partial\Phi}{\partial n}=f(S)$.
On a bounded regular surface S has a unique solution.
A regular surface S has in every point a tangent plane, and curvatures are everywhere continuous.
The problem is: What happens if the surface S degenerates into the two sides of a finite region of the x,y plane?

2. As simple example consider the segment $-1 < x < +1$ in the (x,y) plane where w and Φ are solutions of $\Phi_{xx}+\Phi_{yy}=0$.
Green's function of the second kind is found by conformal mapping.
Solution is explicitly (regular solution)
$$\Phi_P = \frac{1}{2\pi}\int_{-1}^{+1} w^+(x)G_P(x,+0)dx - \frac{1}{2\pi}\int_{-1}^{+1} w^-(x)G_P(x,-0)dx.$$
Obviously Green's function can be added, if we take P either in +1 and -1. Moreover any derivative of G_P in tangential direction with respect to P on the segment ($P\longrightarrow A$ or $P\longrightarrow B$) can be added without affecting the given values of $\frac{\partial\Phi}{\partial n}$ on the segment.

3. <u>Aerodynamical interpretation</u>
Thin aerofoil theory.
Φ = velocity potential of disturbance flow
$\varphi = u = \Phi_x$ = acceleration potential = pressure
regular solution for Φ gives infinite pressure at trailing edge .
regular solution for φ gives no lift.
Add a singular potential, which must be integrable. This corresponds to a pressure dipole at the leading edge.

4. General case

Is it possible to derive the singular solutions for this degenerated surface as limiting case of a regular surface S? Suppose the equation of S is

$$\left\{ z - \varepsilon\, f(x,y) \right\}^2 = \varepsilon^2\, g(x,y)$$

where $g(x,y) > 0$ in a bounded region of the (x,y) plane.
$g=0$ is the edge of the wing.
$z = \varepsilon f(x,y)$ = mean surface, $\varepsilon\sqrt{g}$ = thickness distribution.
Normal vector is

$$\underline{n} = (\varepsilon f_x,\ \varepsilon f_y - 1) \pm \frac{\varepsilon}{2\sqrt{g}}(g_x, g_y, 0).$$

We derive an expression for the normal vector $\underline{n}$ in terms of δ-distributions. Consider a "test vector" $\underline{v} = (u,v,w)$ and the flux

$$\tilde{\Phi} = \int_S (\underline{v}.\underline{n})\, dS \ .$$

Three regions: Upper surface S^+, Lower surface S^-, edge $0 \leqq g \leqq \varepsilon^2$.
Contribution of surface regions to $\tilde{\Phi}$ =

$$\iint_{S^+} \left[-\varepsilon\left(f_x + \frac{g_x}{2\sqrt{g}}\right)u - \varepsilon\left(f_y + \frac{g_y}{2\sqrt{g}}\right)v + w \right] dxdy \ -$$

$$- \iint_{S^-} \left[-\varepsilon\left(f_x - \frac{g_x}{2\sqrt{g}}\right)u - \varepsilon\left(f_y - \frac{g_y}{2\sqrt{g}}\right)v + w \right] dxdy \ .$$

For the edge introduce local coordinates. s along the contour $g=0$, ν normal to contour and σ along a normal cross section.
In this plane the flux is

$$\int (\underline{v}.\underline{n})\, d\sigma \approx \int (\underline{v}.\underline{\nu})\, d\sigma = (\underline{v}.\underline{\nu})(2\varepsilon\sqrt{g})_{\nu = \varepsilon^2}$$

where $\nu = \dfrac{g_x, g_y, 0}{\sqrt{g_x^2 + g_y^2}}$.

Total flux through the edge is

$$\frac{\Phi_e}{\varepsilon} = \oint \left[(\underline{v} \cdot \underline{\nu})(2\sqrt{g})_{\nu=\varepsilon^2} \right] ds = \int_s \int_\nu (\underline{v} \cdot \underline{\nu})\ \delta(\nu) 2\sqrt{g(\nu)} . d\nu\ ds$$

where $\delta(\nu)$ is a δ-distribution.
If $\underline{v}$ is regular, edge region gives no contribution, since $g(\nu) \rightarrow 0$, but if

$$\lim_{\nu \rightarrow 0} (\underline{v} \cdot \underline{\nu}) \sqrt{g(\nu)}$$ is finite, a contribution remains.

Then Neumann's problem reads:

$$\varphi_n = \varphi_z + 2\varepsilon\, \delta(\nu) \ . \left\{ \sqrt{g} . \frac{\partial \varphi}{\partial \underline{\nu}} \right\}$$

and, in addition to the contribution of φ_z on the surface an additional singular potential of streigth $\frac{\partial \varphi}{\partial \underline{\nu}}$ must be added. This term can also be interpreted as a dipole by some transformations $\delta(\nu) \sqrt{g} \frac{\partial \varphi}{\partial \nu} = - \delta'(\nu) \sqrt{g} . \lambda(s)$.

Numerical Analysis of a supersonic flow with a detached shock

by

P.R. Garabedian

We consider the flow behind a curved shock wave possessing axial symmetry. The flow in front of the shock is assumed to be uniform, whereas that behind is determined by a quasi-linear partial differential equation which is elliptic in a subsonic region and hyperbolic in a supersonic region. We suppose that the shock wave is given and treat the inverse problem of locating the body which generates it. Because of the nature of the shock conditions, this leads to a Cauchy problem for the above partial differential equation, with the shock as the initial line. The interest in this situation stems from the appearance of a subsonic, or elliptic, region in which Cauchy's problem in the usual sense is not properly set. This difficulty is overcome by a continuation of the solution of the partial differential equation into the complex domain. In this manner, a class of shock curves can be found within which Cauchy's problem is properly set, even in the elliptic case. A stable numerical scheme for the machine computation of flows of the above type results from the present analysis. Examples calculated by H.M. Lieberstein and the present writer will be presented.

Boundary value problems and trigonometrical expansions

by

H.A. Lauwerier

Summary

The following boundary value problem will be considered.

$$
\begin{array}{lll}
0 < x < \pi, \quad y > 0 & (\Delta - q^2)\, F(x,y) = 0, & \\
y \longrightarrow \infty & F = 0, & (1) \\
x = 0, \quad x = \pi & F = 0, & \\
0 < x < \pi, \quad y = 0 & \cos\frac{\alpha\pi}{2}\frac{\partial F}{\partial y} + \sin\frac{\alpha\pi}{2}\frac{\partial F}{\partial x} + f(x) = 0, &
\end{array}
$$

where $q \geq 0$, $0 \leq \alpha \leq 1$ and $f(x)$ is a given function.

This problem may be reduced either to a singular integral equation or to a trigonometrical expansion of the following kind

$$f(x) = \sum_{1}^{\infty} b_k\,(\sin kx - \theta_k \cos kx), \quad 0<x<\pi, \qquad (2)$$

where the "phases" θ_k are given.

In the special case q=0 we have $\theta_k = \mathrm{tg}\,\frac{\alpha\pi}{2}$ for all k. Then the expansion problem (2) is explicitly soluble. The convergence of (2) is generally of subharmonic order $b_k = O(k^{-1+\alpha})$. If, however,

$$\int_0^{\pi} \left(\mathrm{tg}\,\frac{x}{2}\right)^{\alpha-1} f(x)dx = 0 \qquad (3)$$

the convergence is of hyperharmonic order, $b_k = O(k^{-1-\alpha})$.

The problem (1) has for q=0 a solution which is uniformly bounded in the given region. The first partial derivatives are infinite of order α at (0,0) and finite at $(\pi,0)$.

In the case $q \neq 0$ we have $\theta_k = \theta + O(k^{-2})$ for $k \longrightarrow \infty$. By means of the solution of (2) for $\theta_k = \theta$ the expansion (2) can be reduced to an ordinary integral equation of the Fredholm type. In this case similar conclusions can be drawn as for q=0. In particular (3) should be replaced by

$$\int_0^{\pi} m(x)f(x)dx = 0 \qquad (4)$$

where m(x) is determined by

$$\int_0^{\pi} m(x)\,(\sin kx - \theta_k \cos kx)dx = 0 \text{ for } k \geq 1 \qquad (5)$$